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The Korteweg-deVries and the Kadomtsev-Petviashvili (KP) equations both model the evolution of relatively long water waves of moderate amplitude as they propagate in shallow water. Both equations are completely integrable. In this paper we review the derivation of each equation as an approximate model of shallow water waves, and compare their solutions with some of the experimental observations of waves in shallow water. We also describe in detail the family of doubly periodic KP solutions. These are the natural two-dimensional generalizations of cnoidal waves in one-dimension; one may think of them as describing "typical" (cont'd)

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patterns of non-linear, two-dimensional waves in shallow water.

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Tech Memo #83-3

Integrable Models of Shallow Water Waves

by

Harvey Segur*, Allan Finkel**,
and Hilda Philander*

Abstract:

The Korteweg-de Vries and the Kadomtsev-Petviashvili (KP) equations both model the evolution of relatively long water waves of moderate amplitude as they propagate in shallow water. Both equations are completely integrable. In this paper we review the derivation of each equation as an approximate model of shallow water waves, and compare their solutions with some of the experimental observations of waves in shallow water. We also describe in detail the family of doubly periodic KP solutions. These are the natural two-dimensional generalizations of cnoidal waves in one-dimension; one may think of them as describing "typical" patterns of nonlinear, two-dimensional waves in shallow water.

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*Aeronautical Research Associates of Princeton, P.O. Box 2229, Princeton, NJ 08540

**Institute for Advanced Study, Princeton, NJ 08540

I. Introduction

Two equations that have been studied intensively in recent years are the Korteweg-de Vries (KdV; 1895) equation,

$$u_\tau + 6uu_\chi + u_{\chi\chi\chi} = 0, \tag{1}$$

and a generalization of it due to Kadomtsev and Petviashvili (KP; 1970),

$$(u_\tau + 6uu_\chi + u_{\chi\chi\chi})_\chi + 3u_{\eta\eta} = 0. \tag{2}$$

Most of this interest is due to the remarkable fact that each equation can be solved exactly as an initial-value problem by a method now known as the Inverse Scattering Transform. This method was first discovered for the KdV equation on $-\infty < \chi < \infty$, in the famous papers of Gardner, Greene, Kruskal and Miura (1967, 1974). The corresponding work for (1) on a periodic interval was published by several people during 1974-1976 [Novikov (1974), Dubrovin and Novikov (1974), Dubrovin (1975), Lax (1975), Its and Matveev (1975), McKean and van Moerbeke (1975), McKean and Trubowitz (1976), Dubrovin, Matveev and Novikov (1976)]. For the KP equation on $-\infty < \chi, \eta < \infty$, a method of solution was given very recently by Ablowitz, Bar Yaacov and Fokas (1982; cf. Ablowitz and Fokas, these Proceedings).

As we shall see below, it happens that both (1) and (2) also model the evolution of water waves of moderate amplitude as they propagate in one direction in relatively shallow water. In physical terms, the KdV equation arises if the waves are strictly one-dimensional (i.e.,one spatial dimension plus time), while the KP equation arises if they are only nearly one-dimensional. Because both equations are completely integrable, they provide very precise predictions about the evolution of water waves under appropriate conditions.

This paper has two objectives. The first is to examine the validity of (1) and (2) as models of water waves, by comparing their solutions with some of the available experimental data. The comparison given here will be brief,

but much more detailed verifications of (1) have been made elsewhere (e.g., Hammack and Segur, 1974, 1978). The second objective is to describe in detail a special family of solutions of (2), which are periodic in each of two independent variables. It appears that these doubly-periodic solutions may have great practical importance as models of long water waves. Among other things, they describe: (i) the nonlinear interaction of two trains of finite amplitude waves in shallow water; (ii) the reflection of a train of finite-amplitude shallow-water waves from a vertical wall (by replacing an appropriate line of symmetry with the wall); (iii) the reflection of such a wave train by a change in bottom topography; or (iv) "typical" finite-amplitude, short-crested waves in shallow water. In this last sense, they are the natural generalizations to two dimensions of cnoidal waves in one dimension.

II. Derivation of the Equations

The classical problem of water waves is to find the irrotational motion of an inviscid, incompressible, homogeneous fluid, subject to a constant gravitational force (g). The fluid rests on a horizontal and impermeable bed of infinite extent at $z = -h$ and has a free surface at $z = \zeta(x,y,t)$; see Figure 1. In this derivation we neglect the effects of surface tension at the free surface, although it can be included without difficulty (e.g., see Ch. 4 of Ablowitz and Segur, 1981).

INSERT FIGURE 1 ABOUT HERE

The fluid has a velocity potential, ϕ, which satisfies

$$\nabla^2 \phi = 0, \qquad -h<z<\zeta(x,y,t); \tag{3}$$

(irrotational motion of an incompressible fluid). It is subject to boundary conditions on the bottom, $z = -h$:

$$\phi_z = 0, \tag{4}$$

(impermeable bed); and along the free surface, $z = \zeta$:

$$\frac{D\zeta}{Dt} \equiv \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y = \phi_z \tag{5}$$

(kinematic condition);

$$\phi_t + g\zeta + \frac{1}{2}\left|\nabla\phi\right|^2 = 0 \tag{6}$$

(dynamic condition).

Boundary conditions in (x,y) and initial conditions also are required. If the waves in question are isolated, then $\nabla\phi$ and ζ should vanish as $(x^2 + y^2)\to\infty$. In other problems, periodic boundary conditions in x and in y may be relevant.

This problem, first posed by Stokes (1847), remains unsolved. To make further progress, we impose additional assumptions on the solutions of (3)-(6). The first such assumption is that the wave amplitudes should be small. If we interpret "small" to mean infinitesimal, then we may linearize (3)-(6) about $\nabla\phi = 0$, $\zeta = 0$, and seek solutions of the linearized equations proportional to $\exp\{i(kx+my-\omega t)\}$ (e.g., see Lamb, 1932, §§ 228, 266, 267). The result is the linearized dispersion relation,

$$\omega^2 = g\kappa \tanh \kappa h \quad , \tag{7}$$

where $\kappa^2 = k^2+m^2$. From this one computes the group velocity and shows that the linearized problem is dispersive at most wave numbers, but not as $\kappa\to 0$, (i.e., long waves, or shallow water waves), where it is only weakly dispersive. Both (1) and (2) arise as models of the water wave problem in this weakly dispersive limit, $\kappa h \ll 1$.

To derive (1) or (2), we assume that:

(A) wave amplitudes are small,

$$\varepsilon \equiv \zeta_{max}/h \ll 1;$$

(B) the relevant length scale in the x-direction is much longer than the fluid depth (i.e., shallow water waves),

$$(kh)^2 \ll 1;$$

(C) either the motion is strictly one-dimensional (for KdV),

$$m = 0,$$

(C') or it is nearly one-dimensional (for KP)

$$\left(\frac{m}{k}\right)^2 \ll 1;$$

(D) All of these effects balance,

$$\text{KdV:} \quad (kh)^2 = O(\varepsilon), \qquad \text{KP:} \quad (kh)^2 = O(\varepsilon) = O(mh).$$

These assumptions imply a certain scaling of the original equations, (see (9) below), which may then be solved as a perturbation series, term-by-term in ε. [We omit the details here, but the entire derivation may be found in Chapter 4 of Ablowitz and Segur (1981), or elsewhere.]

At leading order, the equations are linear (from (A)), nondispersive (from (B)), and one-dimensional (from (C)). The solution is simply the solution of the linear, one-dimensional wave equation:

$$\zeta\,(x,y,t) = \varepsilon h\,[f(x-\sqrt{gh}\;t;y) + F(x+\sqrt{gh}\;t;y)] + O(\varepsilon^2). \tag{8}$$

At this order, every wave has permanent form, not because it is a soliton but because we are solving the one-dimensional, linear wave equation.

When this perturbation expansion is carried to second order, the one-dimensional, linear wave equation has homogeneous (forcing) terms representing weak nonlinearity, weak dispersion and weak two-dimensionality. Each of these effects contributes to a secular term at second order. We may eliminate them by introducing a second slow time-scale (T):

$$r = \sqrt{\varepsilon}\ (x-\sqrt{gh}\ t)/h, \qquad l = \sqrt{\varepsilon}\ (x+\sqrt{gh}\ t)/h,$$

$$\eta = \varepsilon y/h, \qquad T = \left(\varepsilon\right)^{3/2} \sqrt{g/h}\ t, \tag{9}$$

and requiring that the right-running waves satisfy

$$(2\ f_T + 3ff_r + \frac{1}{3}\ f_{rrr})_r + f_{\eta\eta} = 0, \tag{10}$$

which may be rescaled to (2). In the one-dimensional case, $\partial_\eta \equiv 0$, and (10) leads to (1). The left-running waves must satisfy a similar equation.

At this order, no secular terms arise from interactions between left- and right-running waves provided that f and F are smooth, and that $\int f dr$, $\int F dl$ remain bounded. For periodic initial data, the latter requirement implies that

$$\oint f dr = 0, \qquad \oint F dl = 0, \tag{11}$$

where the integrals are taken over one period. Because the original equations are galilean-invariant, (11) amounts only to a normalization.

Some observations about the physical meaning of (1) and (2) may be made at this point.

(A) The time in (1) or (2) has the physical meaning of a slow time-scale. Concepts like solitons that are implied by these equations have physical meaning only on this long time-scale.

(B) Our original equations, (3) - (6), are themselves only approximately correct, because they neglect real effects like viscosity. Thus, there is a short (linear, non-dispersive, one-dimensional), time-scale, on which all waves have permanent form, from (8). This is followed by a longer (KdV or KP) time-scale on which soliton interactions are relevant. This is then followed by an even longer time-scale on which other effects (like viscosity) become important.

(C) The KdV and KP equations are fully nonlinear, but they model water waves only when the water waves are weakly nonlinear. Fully nonlinear water waves are known to break, as solutions of (1) or (2) do not. Similarly, (2) models water waves that are only weakly two-dimensional, although this restriction is not evident in (2) by itself.

(D) Because (1) and (2) are first-order in time, neither describes the interactions of left- with right-running waves. However, this is not because such interactions were not admitted, but because they are not important on the time-scale on which these equations apply. Their interaction is given by (8), to leading order.

III. Experimental Evidence of Solitons

The consequences of the theory of Inverse Scattering Transforms for the KdV equation are as follows. (Details may be found in Ablowitz and Segur, 1981, among other places.) Let $u_0(\chi)$ be any smooth function on $(-\infty,\infty)$ that vanishes rapidly along with its derivatives as $\chi\to\pm\infty$. Then (1) has a unique solution that coincides with $u_0(\chi)$ at $\tau=0$. As $\tau\to\infty$, this solution evolves into N isolated solitons, ordered by amplitude, followed by an oscillatory wave-train that disperses in time ("radiation"). When the solitons have separated, each is given (locally) by

$$u(\chi,\tau) = 2k^2 \operatorname{sech}^2 \{k(\chi - 4k^2\tau + \chi_0)\}. \tag{12}$$

The number of solitons (N), their amplitudes (k_j, j=1, ..., N), and all the other details of the long-time solution can be found directly from $u_0(\chi)$. There is no need to advance (1) numerically in time.

Corresponding to (12) is a water wave, whose surface elevation is given by

$$\zeta(x,t) = \frac{4}{3}\varepsilon k^2 h \operatorname{sech}^2 \left\{ \frac{\sqrt{\varepsilon}k}{h} \left[x \pm \sqrt{gh}\left(1 + \frac{2\varepsilon k^2}{3}\right)t + x_0 \right] \right\} + O(\varepsilon^2). \tag{13}$$

Similarly, all the other consequences of (1) imply predictions about real water waves. Hammack and Segur (1974, 1978) tested several aspects of this theory by comparing it with laboratory experiments. Next we briefly reiterate some of their results, to give the reader an idea of the validity of the KdV equation as a model of long water waves of moderate amplitude. (Experimental comparisons of (1) with water waves also have been made by Zabusky and Galvin (1971), and by Weidman and Maxworthy (1978).)

INSERT FIGURE 2 HERE OR BELOW.

The experiments of Hammack and Segur were conducted in a wave tank 31.6 m. long, 61 cm. deep and 39.4 cm. wide. As shown schematically in Figure 2, the wave generator consisted of a rectangular piston located in the tank bed adjacent to the upstream end wall of the tank. The piston spanned the tank width, and was 61 cm. long for the experiments we will discuss. The time-history of its vertical displacement was prescribed for each experiment.

Wave measurements were made during each experiment at several positions down the tank using parallel-wire resistance gauges. In the experiments described here the fluid depth (h) was 5 cm., and the waves were measured at x/h = 0, 20, 180 and 400, where x = 0 at the downstream edge of the piston.

Figure 3 shows a wave generated simply by raising the piston. The piston motion was fast enough that the shape of the wave at $x/h = 0$ is effectively the shape of the piston; (because of the reflecting wall at the upstream end of the piston, the wave at $x = 0$ was actually twice as long as the piston and half as high as its displacement).

INSERT FIGURE 3 SOMEWHERE NEAR HERE.

On a short time-scale, according to (8), this wave should simply translate with speed $\sqrt{gh}$. The wave measured at $x/h = 20$ (Figure 3b) fits this description approximately; its shape is basically that of the wave at $x = 0$. (The front of the wave is to the left in these figures, and a wave which translates with speed $\sqrt{gh}$ shows no horizontal displacement in succeeding frames).

That solitons emerge on a long time scale may be seen in Figures 3c, d. Solving the appropriate scattering problem with the wave measured at $x = 0$ as the potential yields 3 discrete eigenvalues, representing 3 solitons. These correspond to the 3 positive, more-or-less permanent waves seen at $x/h = 180$ and $x/h = 400$. According to (13), these waves all should move to the left in these figures, since their speeds all exceed $\sqrt{gh}$. That they do not is a measure of the effect of viscosity in these experiments.

Even so, we assert that these waves are solitons on the basis of their shapes. The entire profile of a single soliton is determined from (13) once its amplitude is known. The peak amplitudes of the first two waves in Figure 3d were measured and the dots in that figure represent evaluations of (13) based on those amplitudes. The agreement with the measured wave shapes is striking.

The results shown in Figure 3 suggest the following picture of long water waves of moderate amplitude.

a) There is a short (linear) time-scale, during which the left- and right-running waves separate from each other.

b) There is a long (KdV) time-scale, during which the right- (or left-) running waves evolve in N solitons plus radiation.

c) There is an even longer viscous time-scale, during which the energy in these solitons is gradually dissipated. Because the KdV time-scale is shorter, however, the solitons continually readjust their shapes and speeds as they lose energy so that locally, as in Figure 3d, they look and act like solitons.

Several other experiments were performed in this series, to test other aspects of the KdV theory. We refer the reader to the original papers or to Chapter 4 of Ablowitz and Segur (1981) for more details.

Next we consider the KP equation, (2), as a model of nearly one-dimensional water waves of moderate amplitude, propagating in shallow water. Here the theory is still incomplete, and we are aware of no systematic experimental study. Consequently we are forced to discuss special solutions of (2), and fortuitous experimental observations.

Obviously, every KdV solution also solves KP. More generally, Satsuma (1976) showed that (2) admits an N-soliton solution, with the N solitons traveling in N different directions. The two-soliton formula is

$$u(\chi,\eta,\tau) = 2\, \partial_\chi^2 \, \ln f, \tag{14}$$

where

$$f = 1 + \exp(\phi_1) + \exp(\phi_2) + \exp(\phi_1+\phi_2+A),$$

$$\phi_j = k_j\, (\chi + p_j\eta - c_j\tau), \qquad c_j = k_j^2 + 3p_j^2 \quad ,$$

$$\exp(A) = \frac{(k_1-k_2)^2 - (p_1-p_2)^2}{(k_1+k_2)^2 - (p_1-p_2)^2} \quad .$$

A typical solution is shown in Figure 4. Far from the interaction region, each wave is essentially a KdV soliton, but traveling at an angle to the χ-axis. The interaction of the two waves is necessarily nonlinear, and a phase shift of each wave as a result of the interaction is evident. There are two phases in this solution and two spatial coordinates, so unless $p_1=p_2$ in (14), there is a uniformly translating coordinate system in which this solution is stationary.

FIGURES 4 & 5 GO ON THE SAME PAGE, SOMEWHERE NEAR HERE

How well does this two-soliton solution of (2) predict water wave interactions? No quantitative experimental data is available but the photograph in Figure 5, of two long-crested waves interacting in shallow water off a beach in Oregon, certainly is suggestive. Each of the two waves apparently is part of a train of periodic waves coming in from deep water, but their wavelengths seem to be long enough that each acts like a solitary wave in shallow water. The comparison of Figures 4 and 5 is only qualitative, of course, but certainly it suggests that the KP equation might be as useful for weakly two-dimensional water waves as the KdV equation is for one-dimensional waves.

Because (2) only models weakly two-dimensional wave interactions, some oblique interactions are not predicted by it. Miles (1977a,b) examined oblique interactions of two solitary water waves of moderate amplitude, using a method based directly on (3)-(6) and which includes (14), the two-soliton solution of (2), as a special case. He found two possible types of interactions, depending on whether or not two-dimensionality dominated nonlinearity. If the angle between the two waves were large enough, each segment of one wave is affected by the other only for a short time, and the interaction is "weak". These interactions are qualitatively like linear wave interactions, as shown in Figure 6a. They correspond to the limit $(p_1-p_2)^2 \gg (k_1 \pm k_2)^2$ in (14). Alternatively, when the angle between the waves in small, then each segment of one wave feels the other wave for a long time, and the interaction is "strong". Strong interactions are shown in Figures 4, 5, and 6b.

FIGURE 6 GOES NEXT TO THIS PARAGRAPH SOMEWHERE

In the water wave problem, these strong interactions occur only if the angle between the two waves does not exceed a certain critical value. Maxworthy (1980) found experimental evidence that such a critical angle exists, and that the interaction of two solitary waves of given amplitude changes from "weak" to "strong" as one moves through this critical angle. Johnson (1982) has examined this critical angle in more detail.

IV. Periodic Waves in Shallow Water

We turn now to the question of periodic waves of moderate amplitude in shallow water, and to periodic solutions of (1) and (2). The derivation of (1) or (2) given in §II remains valid for periodic initial data, provided that the waves still satisfy assumptions (A)-(D) and the normalization condition, (11). In the case of one-dimensional periodic waves, $f(r)$ and $F(\ell)$ in (8) are periodic functions, and (1) becomes the appropriate evolution equation on $0 < \chi < L$, with periodic boundary conditions,

$$u(\chi + L,\tau) = u(\chi,\tau), \tag{15a}$$

$$\int_0^L d\chi\; u = 0. \tag{15b}$$

The first non-trivial solution of the periodic KdV problem was given by Korteweg and de Vries (1895):

$$u(\chi,\tau) = 2k^2\nu^2\, \mathrm{cn}^2\, [k(\chi-c\tau) + \chi_0;\, \nu] + u_0. \tag{16}$$

Here $\mathrm{cn}\,[\phi;\nu]$ is a Jacobian elliptic function with modulus ν $(0 < \nu < 1)$, whence the name "cnoidal wave"; also,

$$c = 6u_0 - 4k^2\,(1-2\nu),$$

$$kL = 2K(\nu),$$

$$u_0 = -2k^2\left[\frac{E(\nu)}{K(\nu)} - 1 + \nu^2\right],$$

where $K(\nu)$, $E(\nu)$ are the complete elliptic integrals of the first and second kinds, respectively (cf. Byrd and Friedman, 1971). A typical cnoidal wave is shown in Figure 7. These waves reduce to infinitesimal, sinusoidal waves if $\nu \to 0$, and to solitons, (12), if $\nu \to 1$.

FIGURE 7 NEAR HERE

From one perspective, we may view cnoidal waves as the simplest solutions of the KdV problem with periodic boundary conditions. Viewed from a more physical perspective, (16) represents a periodic, shallow-water wave according to

$$\zeta(x,t) = \frac{2}{3}\varepsilon h u\ (\chi,\tau) + O(\varepsilon^2).$$

Thus, (16) defines a family of "typical" nonlinear waves in shallow water, from which one may extract "typical" values of wave speeds, forces, etc., for engineering purposes. This second viewpoint is common among ocean engineers, naval architects and others responsible for the design of large structures in relatively shallow water (e.g., see the book on the engineering design of such structures by Sarpkaya and Isaacson, 1981).

The practical value of this approach is evident: it provides realistic estimates of forces from waves of finite amplitude. Its major limitation also is evident: it is a one-dimensional theory. Water waves which impinge on a vertical wall obliquely, (i.e., waves coming from almost any direction) are excluded from consideration. In particular, if a wave impinges on a wall of some structure obliquely, the point of interaction moves along the wall at a finite speed, and could conceivably excite a reasonant frequency of the structure. Such wave-structure interactions cannot be predicted using only

cnoidal waves, because the one-dimensional nature of cnoidal waves excludes this possibility.

Thus there is a practical need for a two-dimensional generalization of cnoidal waves. We discuss here the simplest meaningful generalization: doubly periodic solutions of the KP equation. As we will see, these solutions are periodic in each of two real, generally non-orthogonal directions, and except for degenerate cases, they are stationary in a uniformly translating coordinate system.

Without any calculations, one can almost guess how doubly periodic KP solutions might look. Because the cnoidal wave is a periodic generalization of one soliton, one should seek a doubly periodic generalization of a two-soliton solution. Thus, Figure 8 shows a sketch of hypothetical "weak" and "strong" interactions of periodic waves, obtained simply by extending periodically the interactions of solitons shown in Figure 6. Note that with strong interactions, all of the waves are short-crested. As with Figure 6, one expects each of these conjectured interaction figures to be stationary in an appropriately translating coordinate system.

FIGURE 8 & 9 GO ON THE SAME PAGE, NEAR HERE.

Preliminary experiments by Hammack (1980, unpublished) seem to confirm that some obliquely interacting, shallow-water waves form patterns like those shown in Figure 8b. The experiments were performed in a simple ripple tank (6.1 m. long x 1.13 m. wide) with a periodic wave maker across one end of the tank. [In a ripple tank, light shining through shallow water from below is focussed by wave crests and defocussed by troughs. The result is an instantaneous picture of the pattern of water waves in which the wave crests appear as bright lines.] In the experiment in question, a uniform train of one-dimensional, shallow-water waves was reflected obliquely by a uniform step, placed at 30° to the incident waves, as shown in Figure 9a. The water depth changed discontinuously across the step from 1.9 cm. to 6.2 cm. The incident and reflected waves constituted two trains of obliquely interacting, finite-amplitude, shallow-water waves. Figure 9b shows the observed

interaction pattern, in qualitative agreement with that in Figure 8b. Obviously, such a comparison of a hypothetical solution of (2) with a crude qualitative observation of water waves confirms nothing, but it suggests that doubly periodic KP solutions might have practical importance.

Now let us return from this flight of fancy to the question of doubly periodic solutions of (2). Using methods of algebraic geometry, Krichever (1976) first observed that (2) admits quasi-periodic solutions of the form

$$u\ (\chi,\eta,\tau) = 2\ \partial_\chi^2\ \ell n\ \Theta\ (Z_1,\ \ldots,\ Z_N), \tag{17a}$$

where $\Theta\ (Z_1,\ \ldots,\ Z_N)$ denotes Riemann's theta function, and

$$Z_j = U_j\ \chi + V_j\ \eta + W_j\ \tau + Z_{jo}\ ,\qquad j=1,\ \ldots,\ N. \tag{17b}$$

(Technical definition: a quasi-periodic function with N arguments is periodic in each of its arguments separately.) This idea has been pursued in a series of papers by Krichever and Novikov (see Krichever and Novikov, 1980 for a review). However, Riemann theta functions of several arguments are mathematically well-defined but poorly understood, and it has been difficult to extract concrete information from this work to date.

Fortunately, recent work by Dubrovin (1981) on compact Riemann surfaces of low genus (1, 2 and 3) has provided the information necessary to change (17) from an abstract symbol into a computationally effective tool for these low genera. We note that for Riemann theta functions, the genus of the Riemann surface equals the number of arguments in the theta function. Thus, genus 1 gives back the (single-argument) cnoidal waves discussed above; genus 2, corresponding to a theta function of two arguments, gives the intrinsically two-dimensional, doubly periodic generalizations of cnoidal waves that are of interest here. We will not consider solutions of genus 3 in this paper.

Dubrovin's work enables one to solve the following problems:

(I) Parameterize all possible real-valued KP solutions that can be expressed in terms of a theta function of two arguments, as in (17).

(II) Give an effective algorithm to compute these solutions.

(III) Given the instantaneous values (e.g., at $\tau=0$) of a KP solution in the form of (17) with N=2, describe this solution for all time.

Because these problems can be solved, one can solve (2) as an initial value problem, for the very restricted family of solutions related to theta functions of genus 2.

Before describing the solutions of these three problems, we mention earlier work by Nakamura (1979) and Bryant (1982) on finding doubly periodic solutions of KP. Nakamura (1979) also used (17), but without the results of Dubrovin (1981) he was unable to give a complete solution of the problem. Bryant (1982) gave an algorithm to compute approximately specific doubly periodic KP solutions directly in terms of Fourier series. No claim of completeness was made in his work.

(I) Parameterize the KP solutions of genus 2.

Note that (2) has several obvious symmetries: χ-, η- and τ-translations, two coordinate reflections ($\eta\to-\eta$; $\chi\to-\chi$, $\tau\to-\tau$), "coordinate rotation" ($\partial_\chi\to\partial_\chi$, $\partial_\eta\to\partial_\eta + \alpha\partial_\chi$, $\partial_\tau\to\partial_\tau-6\alpha\partial_\eta-3\alpha^2\partial_\chi$), galilean invariance ($u\to u+\beta, \chi\to\chi-6\beta\tau$, $\eta\to\eta$, $\tau\to\tau$) and scaling ($u\to\lambda^{-2}u$, $\chi\to\lambda\chi$, $\eta\to\lambda^2\eta, \tau\to\lambda^3\tau$). All except scaling follow from corresponding symmetries in the original problem, (3)-(6); scaling follows from the arbitrary definition of ε in (9). Each of these symmetries introduces a free parameter into any KP solution. Our objective here is to identify the parameters that characterize the KP solutions of genus 2, beyond these automatic parameters. [According to this way of counting, a single KdV soliton (12) has no free parameters, a cnoidal wave (16) has one free parameter, while the two-soliton solutions of KP in (14) have two free parameters.]

Using methods of algebraic geometry, Dubrovin shows that each KP solution of genus 2 corresponds to a point on the surface of a compact Riemann surface of genus 2. (Topologically, this is a sphere with 2 handles). This surface is identified uniquely by a 2x2 Riemann matrix (a symmetric matrix with negative definite real part); i.e., by three complex parameters. A fourth complex parameter identifies the point on the Riemann surface. Thus each KP solution of genus 2 is identified by four parameters. We will see below that if these four parameters all are real and satisfy certain inequalities, then the resulting KP solution is real.

(II) Calculate KP solutions of genus 2.

$$\text{Let } \underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}, \quad \vec{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{18}$$

where b_{ij} and Z_j are arbitrary complex numbers, $p_j=0$ or 1/2, and m_j are arbitrary real integers. A Riemann theta function is defined by a Fourier series of the form

$$\Theta(\vec{Z}) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \exp\left\{\frac{1}{2}\vec{m}\cdot\underline{\underline{B}}\cdot\vec{m} + \vec{m}\cdot\vec{Z}\right\}, \tag{19a}$$

$$\text{where} \quad \vec{m}\cdot\underline{\underline{B}}\cdot\vec{m} = \sum_{i,j=1}^{2} b_{ij}\, m_i\, m_j \quad , \quad \vec{m}\cdot\vec{Z} = \sum_{j=1}^{2} m_j\, Z_j \quad . \tag{19b}$$

In the present application Z_j is given by (17b), and is pure imaginary. Then $\Theta(\vec{Z})$ is real if $\underline{\underline{B}}$ is a real matrix, which we now assume. $\underline{\underline{B}}$ is negative definite if

$$b_{11} + b_{22} < 0, \tag{20a}$$

$$b_{11}\, b_{22} - b_{12}^2 > 0. \tag{20b}$$

Without loss of generality we may also assume

$$b_{22} < b_{11} < 0. \tag{20c}$$

We want (17) to solve KP, with N=2 and $\Theta(\vec{Z})$ defined by (19). It becomes necessary to define certain "theta-constants", as follows.

$$\hat{\Theta}\,[\vec{p}] = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \exp\,\{(\vec{m}+\vec{p})\cdot\underline{\underline{B}}\cdot(\vec{m}+\vec{p})\} \tag{21a}$$

$$\hat{\Theta}_{ij}\,[\vec{p}] = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} (m_i+p_i)\,(m_j+p_j)\,\exp\,\{(\vec{m}+\vec{p})\cdot\underline{\underline{B}}\cdot(\vec{m}+\vec{p})\} \tag{21b}$$

$$\hat{\Theta}_{ijk\ell}\,[\vec{p}] = \sum_{m_1} \sum_{m_2} (m_i+p_i)\,(m_j+p_j)(m_k+p_k)(m_\ell+p_\ell)\,\exp\,\{(\vec{m}+\vec{p})\cdot\underline{\underline{B}}\cdot(\vec{m}+\vec{p})\} \tag{21c}$$

For fixed (i,j,k,ℓ), each of these is a four-component vector, indexed by the four choices of $\vec{p}$.

Define the 4x4 real matrix $\underline{\underline{D}}$ by

$$\underline{\underline{D}} = (\hat{\Theta}_{11}[\vec{p}]\;,\;\hat{\Theta}_{12}[\vec{p}]\;,\;\hat{\Theta}_{22}[\vec{p}]\;,\;\hat{\Theta}[\vec{p}])\;. \tag{22}$$

The Riemann matrix $\underline{\underline{B}}$ is said to be "indecomposable" if

$$\det\,(\underline{\underline{D}}) \neq 0. \tag{23}$$

This condition assures that the theta function associated with $\underline{\underline{B}}$ through (19) is honestly genus 2, and has not degenerated into genus 1. Beyond (20) and

(23), there are no further restrictions on the real matrix $\underline{\underline{B}}$, which completely characterizes the underlying Riemann surface.

Each KP solution of genus 2 corresponds to a point on the Riemann surface. We now select the point, by selecting (U_1, U_2) in (17b). Because of (23), $\underline{\underline{D}}^{-1}$ exists. Denote the rows of this matrix by

$$\underline{\underline{D}}^{-1} = \begin{pmatrix} d^{11}\,[\vec{p}] \\ d^{12}\,[\vec{p}] \\ d^{22}\,[\vec{p}] \\ d\,[\vec{p}] \end{pmatrix}. \tag{24}$$

For $\vec{U} = (U_1, U_2)$, define

$$\partial_U^4\ \hat{\theta}\,[\vec{p}] = \sum_{i,j,k,\ell=1}^{2} U_i\, U_j\, U_k\, U_\ell\ \hat{\theta}_{ijk\ell}\,[\vec{p}]\ . \tag{25}$$

Also define, for $j \geq i$,

$$Q_{ij}\,(\vec{U}) = \sum_{\vec{p}} d^{ij}\,[\vec{p}]\ \ \partial_U^4\,\hat{\theta}\,[\vec{p}]. \tag{26}$$

Dubrovin shows that with N=2, (17) defines a KP solution if and only if $(\vec{U}, \vec{V}, \vec{W})$ are related by

$$U_1\,W_1 + 4\,Q_{11}\,(\vec{U}) + 3\,V_1^2 = 0,\quad U_2\,W_2 + 4\,Q_{22}\,(\vec{U}) + 3\,V_2^2 = 0,$$

$$U_1\,W_2 + U_2\,W_1 + 4\,Q_{12}\,(\vec{U}) + 6\,V_1\,V_2 = 0. \tag{27}$$

[Note: Dubrovin's $\vec{W}$ differs from ours by a factor of (-4), because of different scalings of (2).] Self-consistency of (27) implies

$$(U_1 V_2 - U_2 V_1)^2 + \frac{4}{3} P (U_1, U_2) = 0, \tag{28a}$$

where

$$P (U_1, U_2) = U_1^2 Q_{22} (\vec{U}) + U_2^2 Q_{11} (\vec{U}) - U_1 U_2 Q_{12} (\vec{U}). \tag{28b}$$

If $U_2 \neq 0$, the scaling invariance and coordinate rotation of (2) permit the normalization,

$$\vec{U} = (s, 1), \qquad \vec{V} = (V_1, 0). \tag{29}$$

Given (29), it is evident from (28a) that the six roots of

$$P (s, 1) = 0$$

give six solutions with $\vec{V}=0$, i.e., genus 2 solutions of the KdV equation. These solutions correspond to the six Weierstrass points of the Riemann surface. To obtain real-valued, two-dimensional solutions of KP, we need

$$P (s, 1) > 0 \quad \text{for real } s. \tag{30}$$

Then it follows from (27) that (17) defines a real-valued KP solution of genus 2 if (Z_1, Z_2) are defined using

$$\vec{U} = i (s, 1), \qquad \vec{V} = i \left(2\sqrt{P(s,1)/3}, 0\right),$$

$$\vec{W} = 4i \left(-sQ_{22} (s, 1) + Q_{12} (s, 1), Q_{22} (s, 1)\right). \tag{31}$$

There is one additional solution, corresponding to $U_2=0$. If

$$Q_{22}\ (1,0) > 0, \tag{32}$$

then (17) defines a real-valued KP solution of genus 2 if $(Z_1,\ Z_2)$ are defined using

$$\vec{U} = i\ (1,0)\ , \qquad \vec{V} = i\ (0,\ 2\sqrt{Q_{22}(1,0)/3})\ ,$$

$$\vec{W} = 4i\ (Q_{11}\ (1,0),\ Q_{12}\ (1,0))\ . \tag{33}$$

This KP solution is also a KdV solution if $Q_{22}\ (1,0)$ vanishes.

The parameter s, which selects the point on the Riemann surface, has a simple geometric interpretation. The function $u(\chi,\eta,\tau)$ is defined on a fundamental period parallelogram, then repeated periodically in two directions (see Figure 8). It is easy to show that the area, A, of this parallelogram satisfies

$$(U_1\ V_2 - U_2\ V_1)^2 = \left(\frac{4\pi^2}{A}\right)^2\ . \tag{34}$$

According to (28a), for fixed scaling of (2), choosing a solution of (30) amounts to choosing the area of the period parallelogram.

To summarize, $\underline{\underline{B}}$ contains three real parameters, which must satisfy (20) and (23). Then s is a real parameter that satisfies (30), or (32) in the special case. Every legitimate choice of these four real parameters produces a KP solution of genus 2, using (17) with either (31) or (33). Naturally, this four-parameter family of solutions may be generalized by using the symmetries of the KP equation discussed in problem (I). All of the solutions with $U_1V_2 \neq U_2V_1$ that are produced in this way are: (i) real-valued; (ii) quasi-periodic functions of two variables $(Z_1,\ Z_2)$; and (iii) stationary in time in some uniformly translating coordinate system. In this sense, they

are the natural generalizations of cnoidal waves to two dimensions. We conjecture that no other KP solutions have these three properties, but we have not proven it.

Figure 10 shows some KP solutions of genus 2, obtained by implementing this algorithm numerically. All of the solutions shown there correspond to the same Riemann matrix: $b_{11}=-1.72$, $b_{12}=1.18$, $b_{22}=-3.55$. A phase shift of these short-crested waves, due to their nonlinear interactions, is evident in the figures.

FIGURE 10 GOES HERE OR ABOVE.

(III) Given the instantaneous values of a KP solution of genus 2, describe this solution for all time.

By hypothesis, $u(\chi,\eta,\tau=0)$ has the form (17) with N=2. Because $u(\chi,\eta,0)$ is given pointwise, we may measure $\vec{U}$ and $\vec{V}$ directly. If $U_1V_2 = U_2V_1$, then the solution in question is actually a KdV solution of genus 2, and it may be described using the well-established theory for the periodic KdV equation (e.g., Dubrovin & Novikov, 1974, or Ch. 2.3 of Ablowitz & Segur, 1981). Thus we need to consider only the case in which

$$U_1V_2 \neq U_2V_1 , \tag{35}$$

so that the solution is non-trivially periodic in two spatial directions.

Every KP solution of genus 2 that satisfies (35) is stationary in a uniformly translating coordinate system, whose velocity is given by $\vec{W}$ in (17b). Once $\vec{W}$ is known, then the initial data in a period parallelogram plus $\vec{W}$ determine the solution for all time. It is not necessary to reconstruct $\underline{\underline{B}}$, the Riemann matrix, for KP solutions of genus 2 that satisfy (35).

Algebraic equations for $\vec{W}$ may be obtained in a variety of ways. The method presented here is valid either if $U_2=0$ or if (U_1/U_2) is rational. This restriction is always satisfied in applications, where (U_1/U_2) is measured only to a finite accuracy. In problem (II), it amounts to requiring that the

parameter s be rational.

If (U_1/U_2) is rational (or if $U_2=0$), then $u(\chi,\eta,0)$ is a strictly periodic function of χ, holding (η,τ) fixed. Denote this χ-period by L. Because $u(\chi,\eta,0)$ has the form (17), it follows that for all (χ_0,η),

$$\int_0^L d\chi \; u(\chi+\chi_0, \eta, 0) = 0. \tag{36}$$

Define

$$\phi(\chi,\eta;\chi_0) = -\frac{1}{L}\int_0^L d\xi \cdot \xi u(\chi+\chi_0+\xi,\eta). \tag{37}$$

ϕ is the unique anti-derivative of u with the same periodicity as u, and with zero mean in χ; i.e., ϕ also satisfies (36), and $\partial_\chi\phi=u$. The corresponding anti-derivative of ϕ may be defined in a similar way.

One equation for $\hat{W}$ may be obtained by multiplying (2) by u, and integrated over a period parallelogram (with area A). The result is

$$\iint_A d\chi d\eta \; [u_\chi u_\tau + 6uu_\chi^2 - u_{\chi\chi}^2 + 3u_\eta^2] = 0 . \tag{38}$$

By hypothesis, $u(\chi,\eta,\tau) = f(Z_1, Z_2)$, where Z_j is given by (17b). It follows that

$$(U_1V_2 - U_2V_1)\, u_\tau = (W_1V_2 - W_2V_1)\, u_\chi + (U_1W_2 - U_2W_1)\, u_\eta \; , \tag{39}$$

so (38) becomes

$$\iint_A d\chi d\eta\, [V_2 u_\chi^2 - U_2\, u_\chi u_\eta]\, W_1 + \iint_A d\chi d\eta\, [U_1 u_\chi u_\eta - V_1 u_\chi^2]\, W_2 +$$

$$(U_1 V_2 - U_2 V_1) \iint_A d\chi d\eta\, [6uu_\chi^2 - u_{\chi\chi}^2 + 3u_\eta^2] = 0 \quad . \tag{40}$$

To obtain a second equation, multiply (2) by that anti-derivative of ϕ (i.e., the second integral of u) with the same periodicity as u and ϕ. The result after integrating over the same period parallelogram is:

$$\iint_A d\chi d\eta\, [V_2 u^2 - U_2 u\phi_\eta]\, W_1 + \iint_A d\chi d\eta\, [U_1 u\phi_\eta - V_1 u^2]\, W_2 +$$

$$(U_1 V_2 - U_2 V_1) \iint_A d\chi d\eta\, [3u^3 - u_\chi^2 + 3\phi_\eta^2] = 0 \quad . \tag{41}$$

Equations (40) and (41) are two linear algebraic equations for (W_1, W_2). Wherever they are linearly independent, their common solution defines (W_1, W_2), which completes the mathematical specification of the KP solutions of genus 2. It is easy to show that (40) and (41) are linearly independent for small enough wave amplitudes, but we have not yet established this property in general. In the cases we have tested numerically, (40) and (41) are independent, and their common solution agrees with that in (31).

More generally, given any initial data, $f(\chi,\eta)$, such that

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L d\chi\, f(\chi,\eta) = 0 \ , \tag{42}$$

one may seek the KP solution of genus 2 that best approximates $f(x,\eta)$ at $\tau = 0$, along with an appropriate measure of goodness of fit. It is evident that such approximation procedures will be necessary if these KP solutions of genus 2 are to become a practical tool in physical problems. However, we leave this and other questions of physical implications of these solutions for a future paper.

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Figure Captions

Figure 1. Physical configuration, showing notation for Eq.'s (3)-(6).

Figure 2. Schematic drawing of the wave generator used by Hammack and Segur (1974).

Figure 3. Evolution of surface waves, measured at four locations downstream of the wave maker. The front of each wave is to the left in the coordinate system used, which translates with speed $\sqrt{gh}$. ———, measured wave profiles; ....., soliton profiles computed using (13) and the measured peak amplitude of each wave.

Figure 4. Two soliton solution of the KP equation, with $k_1 = k_2 = 1$, $p_1 = -p_2 = 4$ in (14).

Figure 5. Oblique interaction of two waves in shallow water, observed off a beach in Oregon. (Photograph courtesy of T. Toedtemeier.)

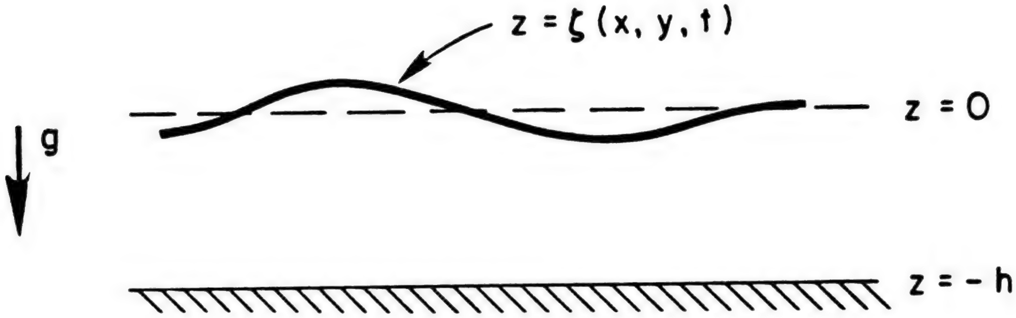
Figure 6. Two types of oblique interactions of KP solitons, showing the patterns of the wave crests. Arrows indicate direction of propagation. (a) Weak interaction, corresponding to a large angle between the waves, or to small amplitude waves; (b) strong interaction, for a small angle between the waves, or to larger amplitude waves.

Figure 7. One period of cnoidal wave solution of the KdV equation, with $\nu^2 = 1/2$ in (16).

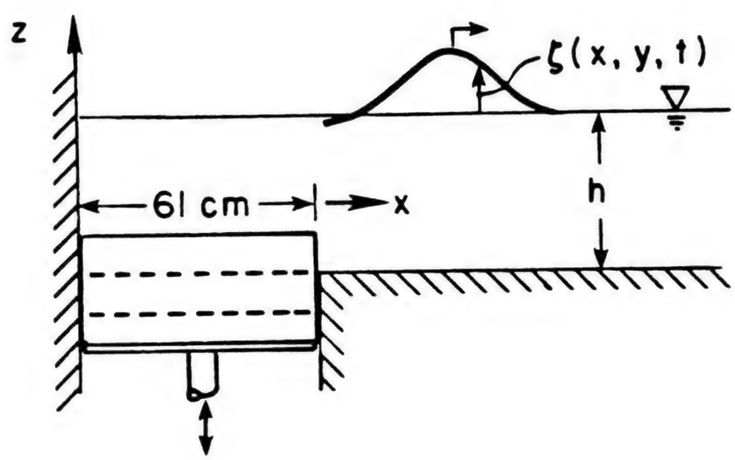
Figure 8. Hypothetical pattern of wave crests for periodic KP solutions. (a) Weak interaction, as in linear theory; (b) strong interaction. The dashed lines indicate a period parallelogram.

Figure 9. (a) Experimental apparatus to create oblique interaction of trains of finite amplitude waves in shallow water. (b) Observed configuration of wave crests. (Photograph courtesy of J.L. Hammack.)

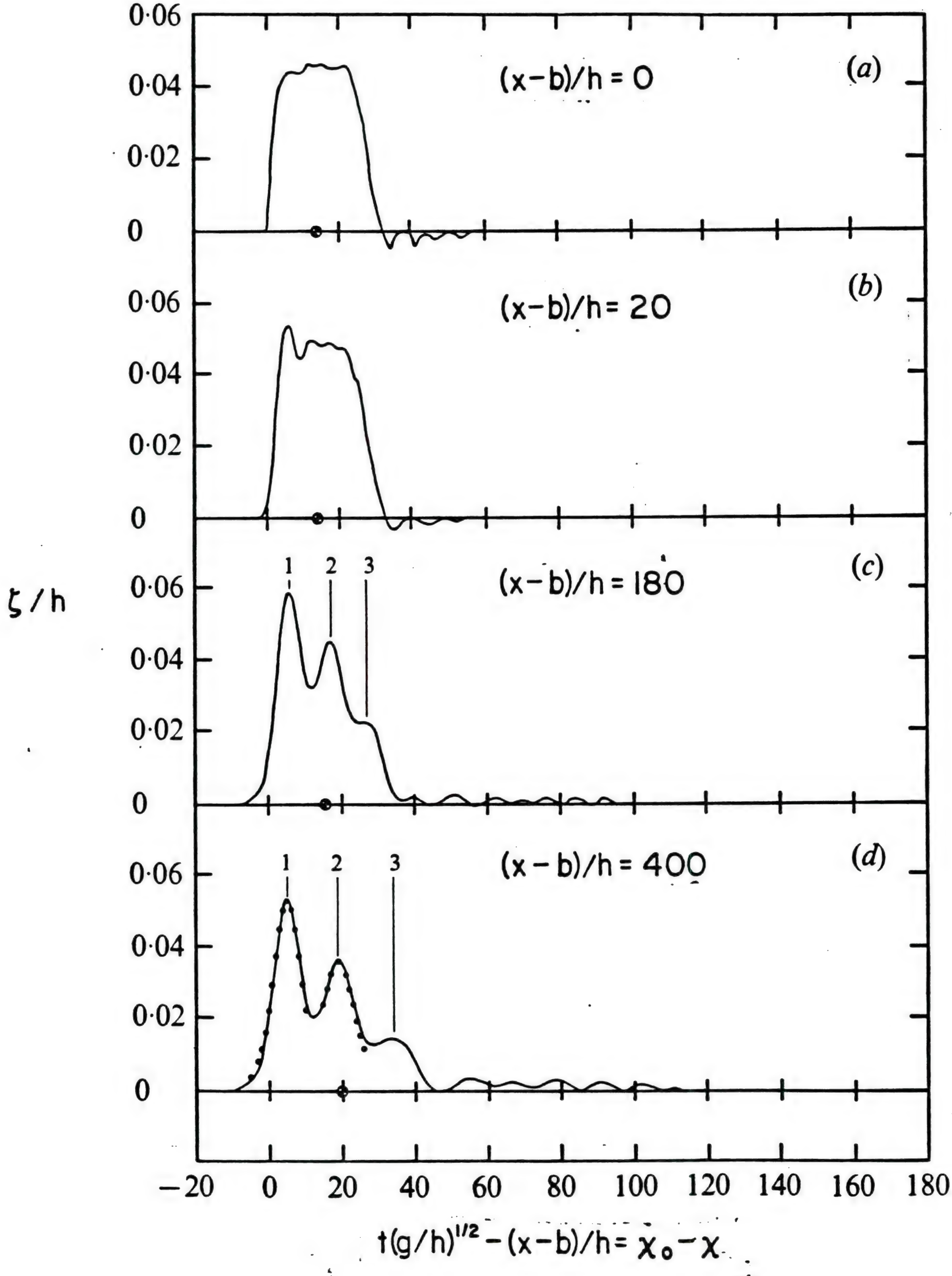
Figure 10. Sample of KP solutions of genus 2, corresponding to different points on the same Riemann surface. (a) $s = U_1/U_2 = 0.0$ in (29); (b) $s = 0.35$, (c) $s = -0.70$.



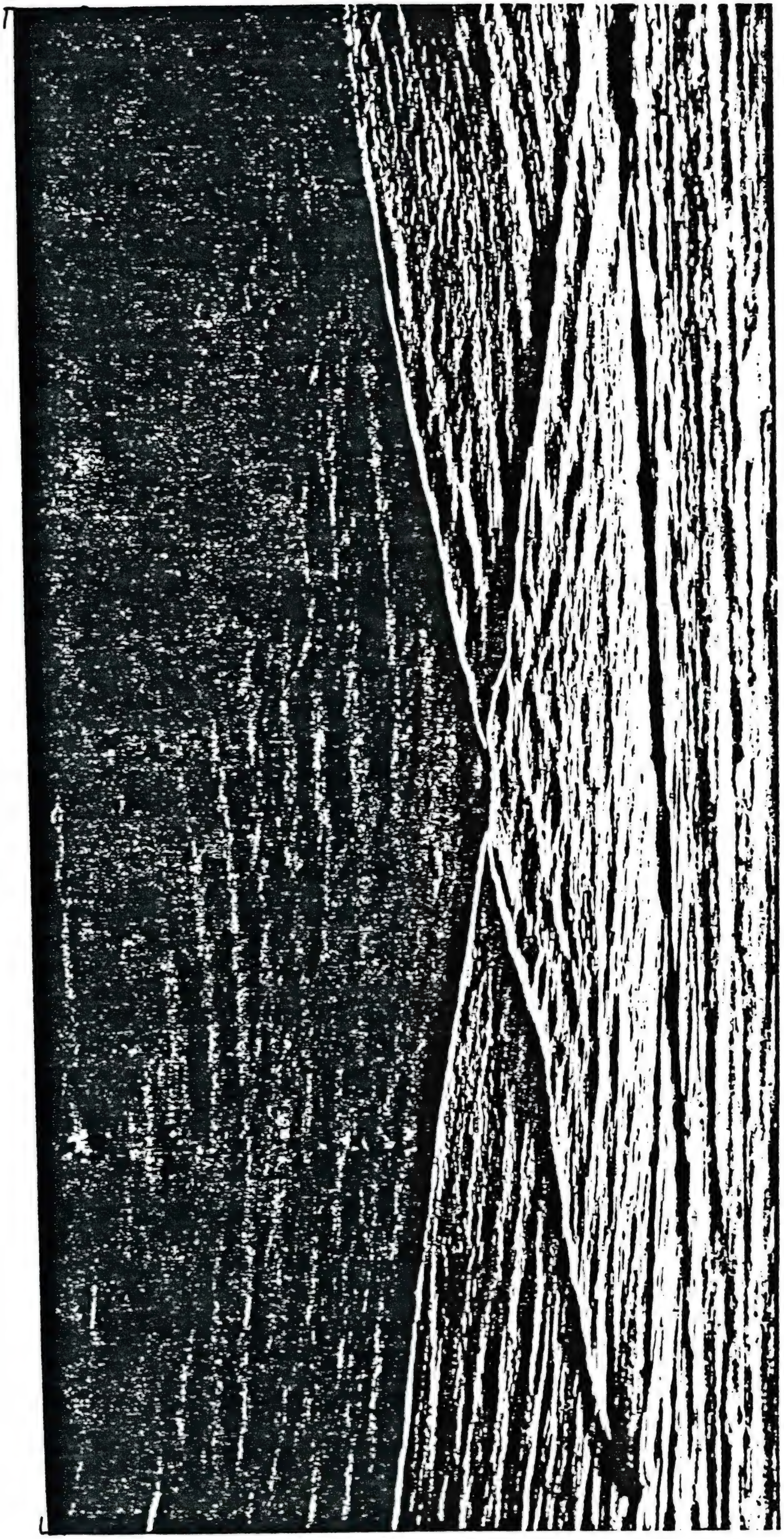
z = ζ (x, y, t)
z = 0
g
z = - h



z
ζ (x, y, t)
61 cm
x
h



(a)
(x−b)/h = 0
(b)
(x−b)/h = 20
(c)
(x−b)/h = 180
1
2
3
(d)
(x−b)/h = 400
1
2
3
ζ / h
0
0·02
0·04
0·06
−20
0
20
40
60
80
100
120
140
160
180
t(g/h)^1/2 − (x−b)/h = χ₀ − χ



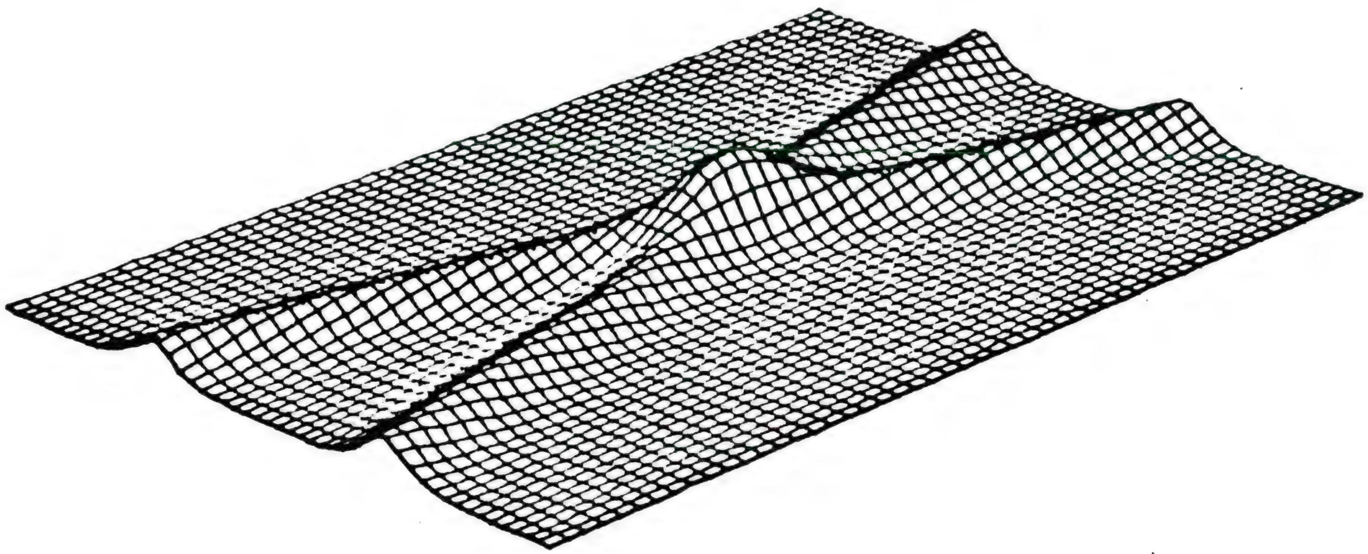
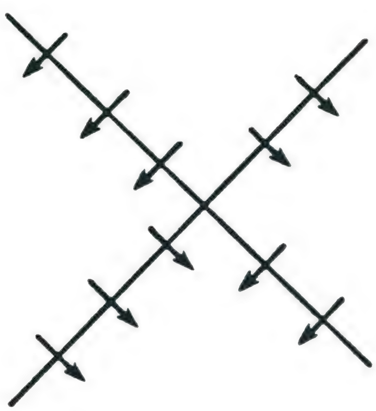


Fig. 7



6a

6b

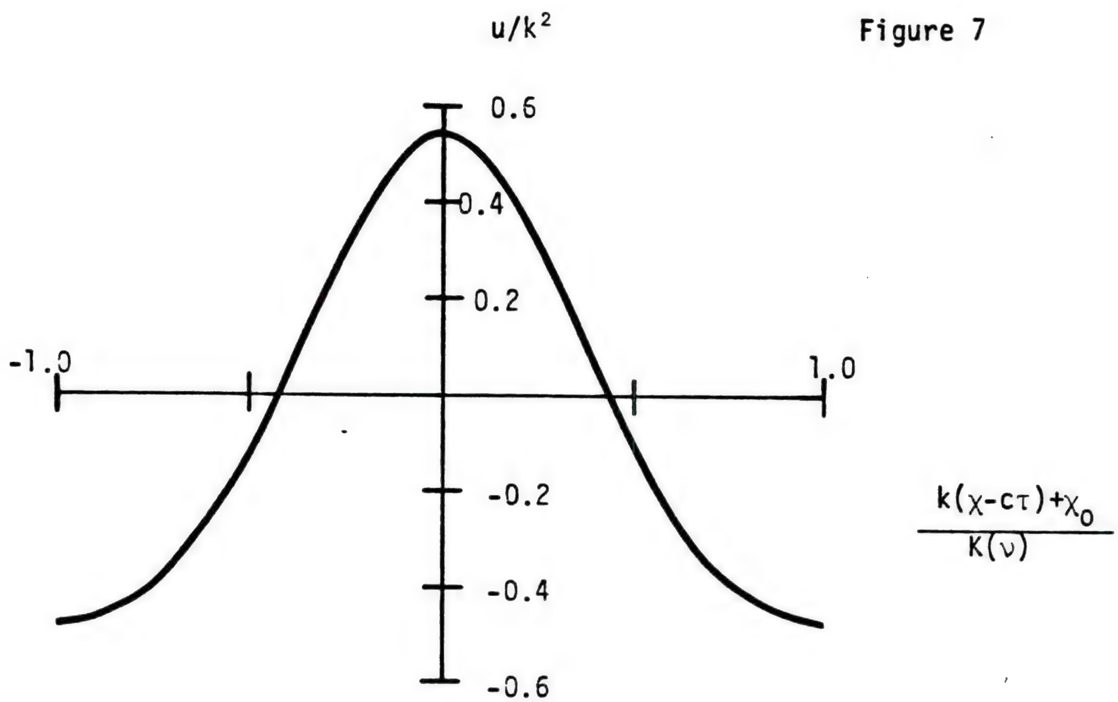
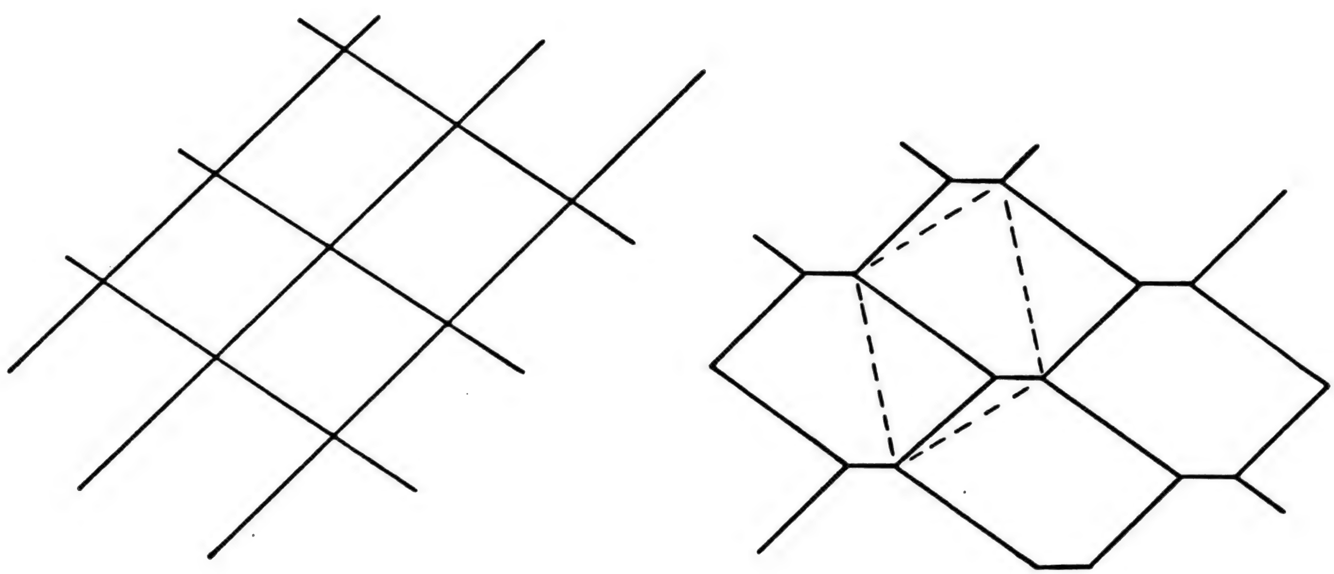


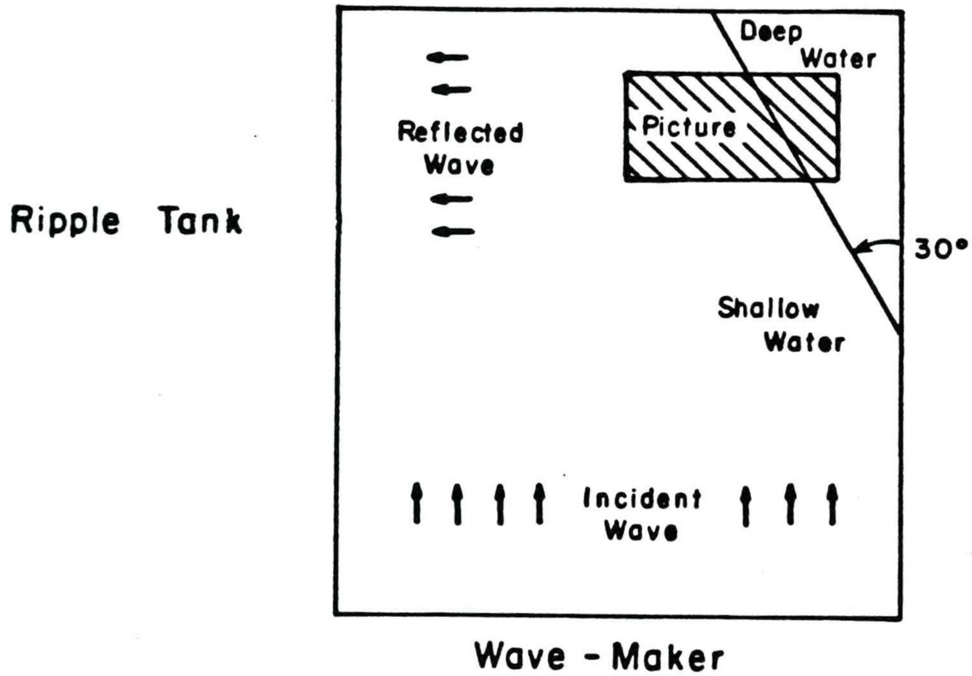
Figure 7



(a)

(b)

Figure 8


Ripple Tank
Deep Water
Picture
Reflected Wave
30°
Shallow Water
Incident Wave
Wave - Maker

← Fig 9a

Fig 9b

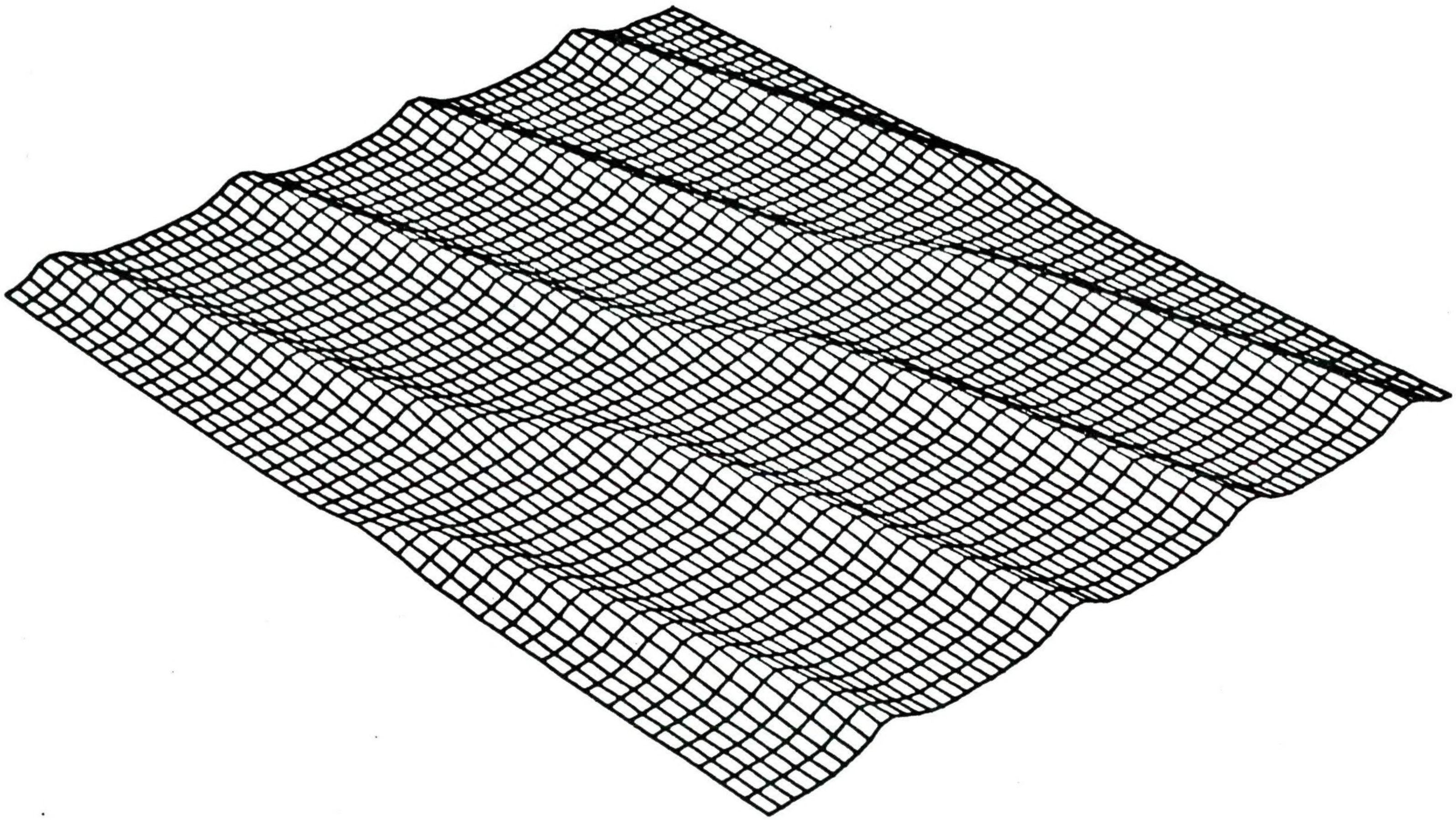


Fig 10a

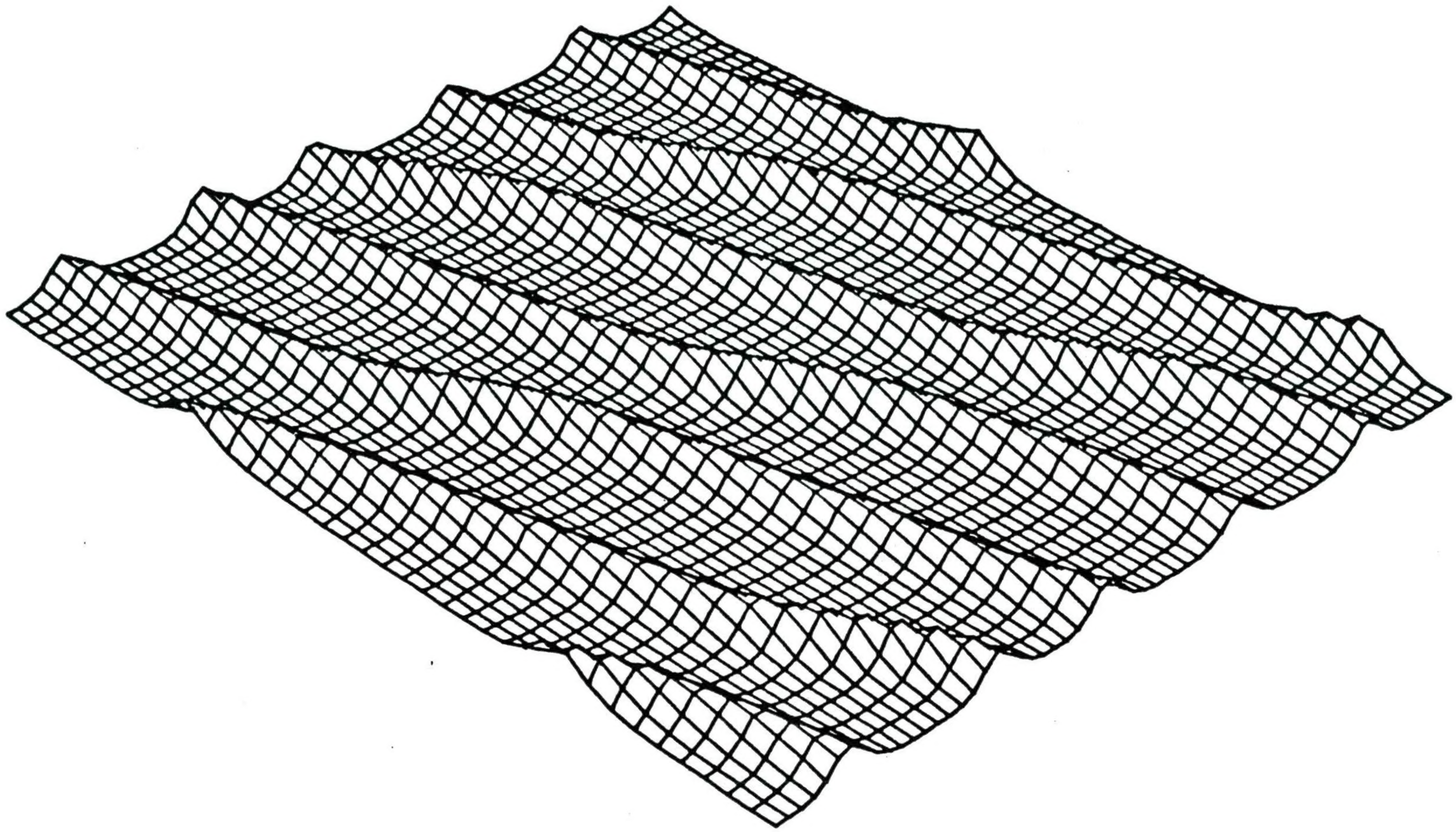


Fig 10b

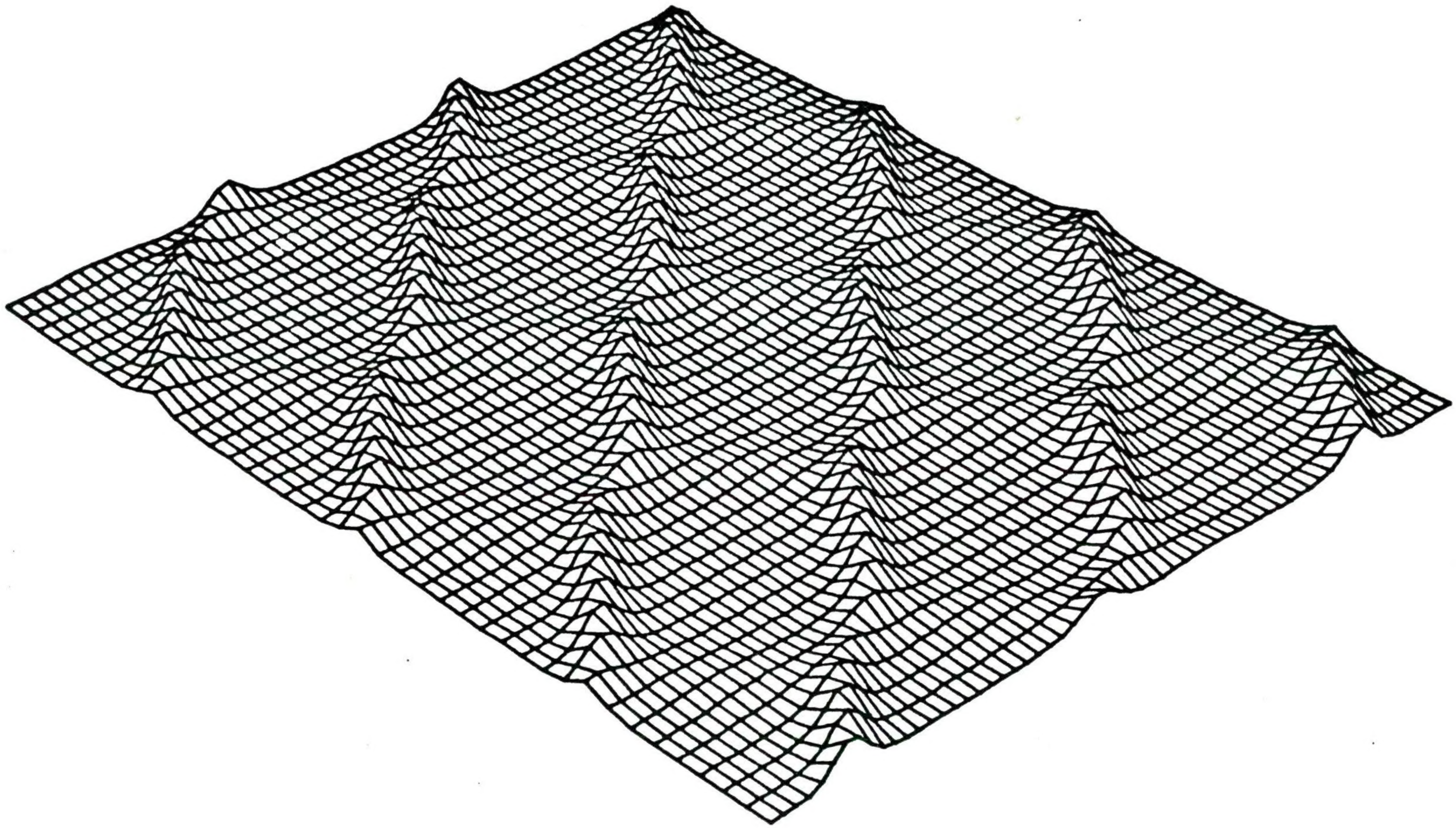


Fig 10 e